A probabilistic position value

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Abstract

In this article, we generalize the position value, defined by Meessen (1988) for the class of deterministic communication situations, to the class of generalized probabilistic communication situations (Gómez et al. (2008)). We provide two characterizations of this new allocation rule. Following in Slikker's (2005a) footsteps, we characterize the probabilistic position value using probabilistic versions of component efficiency and balanced link contributions. Then we generalize the notion of link potential, defined by Slikker (2005b) for the class of deterministic communication situations, to the class of generalized probabilistic communication situations, and use it to characterize our allocation rule. Finally, we show that these two characterizations are logically equivalent.

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1 Introduction

Various economic or social situations in which a group of agents cooperate to achieve a common goal can be appropriately formalized via cooperative games with transferable utility, or TU games. In such games, agents are referred as players. A TU game summarizes all the necessary information concerning the worth

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produced by each coalition of players when they agree to cooperate. It is assumed that any coalition can form. On the other hand, in many situations the collection of possible coalitions is restricted by some social, hierarchical, economical, communicational or technical structures. In this article, we restrict ourselves to the special case of communication situations, introduced by Myerson (1977), in which the cooperation among players is limited because only some undirected and bilateral relations are possible. Then, a communication situation consists of a TU game and a network of possible relations modelled via a graph. The vertices in the graph correspond to the players and the edges correspond to bilateral communication links. In order to measure the impact of restrictions on communication on the gains from cooperation, Myerson (1977) associates to each communication situation a new TU game, the so-called graph-restricted game. The Myerson value of a communication situation is the Shapley value of its graph-restricted game. Myerson (1977, 1980), Borm et al. (1992) and Slikker and van den Nouweland (2001) provide various characterizations of the Myerson value.

Meessen (1988) suggests to associate to each communication situation an alternative TU game that focuses on the role of links: the so-called link game, in which the set of players is the set of links. This link game associates to every set of links the worth produced by the grand coalition of players when the links in this set are the only ones available. The position value, defined by Meessen (1988), shares the Shapley value of each link in the link game equally between its two incident players. The position value of a player is the sum of the gains he collects in this way. Borm et al. (1992) provide a characterizations of the position value that is valid on the class of communication situations in which the graph is cycle-free, whereas Slikker (2005a,b) obtains two new characterizations without the restrictions on the graph existing in the previous one.

In this article, we are interested in situations where the network structure is not given and fixed, i.e. several alternative networks can form and thus players are able to form occasional alliances as well as long term relationships. In this setting, the Myerson approach can be think of as a particular case in which only one network is possible. There are two different ways to model such communication networks.

The first one (Calvo et al. (1999)) consists of considering the communication between two players as a Bernoulli trial. A probabilistic graph maps to each link a probability of realization. These probabilities are considered as independent. A probabilistic communication situation is made up of a TU game and a probabilistic graph on the same set of players. Because of the independence assumption, this approach fails to take into account those situations in which the probability of realization of a set of links is correlated with the realization of another one. For example consider the Airbus' sub-contracting network. Airbus has a stable relationship with firms that produce critical and complex sub-systems, as Thalès Avionics or Latécoère. By way of contrast, for the production of non critical systems, Airbus frequently benchmark its suppliers according to a cost criterion. For instance, Airbus will sub-contract the production of joysticks to SKF with probability p and to Ratier Figeac with probability $1 - p$. These (simplified) relationships can be represented by Figure 1.

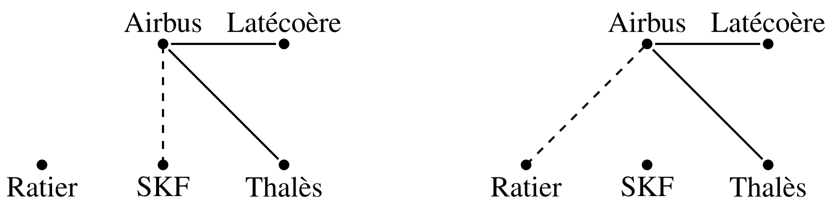

Fig. 1. Airbus' sub-contracting network

As the realization of the link between Airbus and Ratier Figeac is correlated to the realization of the link between Airbus and SKF, this situation cannot be appropriately described using the independence hypothesis.

In the second approach (Gómez et al. (2008)), we dropped the independence assumption and considered the so-called generalized probabilistic graphs, that assign a probability, measuring the likelihood of each one of the potential networks. Restricting the cooperation in a TU game with a generalized probabilistic graph we obtained a generalized probabilistic communication situation. This approach is a generalization of the previous one and extends the range of situations to which can be applied.

The aim of this article is to carry on the work of extension of allocation rules to (generalized) probabilistic communication situations initiated by Calvo et al. (1999) and Gómez et al. (2008). Calvo et al. (1999) define and characterize a natural extension of the Myerson value in terms of component efficiency, fairness and balanced contributions on the class of probabilistic communication situations. Gómez et al. (2008) extend these results to the class of generalized probabilistic communication situations, and describe some properties of the graph-restricted game.

In this article, we extend the definition of the position value to the class of generalized probabilistic communication situations and provide two characterizations of this new allocation rule. Following in Slikker's (2005a) footsteps, we characterize the probabilistic position value using probabilistic versions of component efficiency and balanced link contributions.

Slikker (2005b) also characterizes the position value using a link potential for communication situations that extends the potential for TU games of Hart and Mas-Colell (1989) in a natural way. This link potential focuses on the marginal contributions of a player's link. We will extend too this characterization to the class of generalized probabilistic communication situations.

As Thomson (2001) argues, analysing the logical relations between the axioms permits to highlight their relative power. We show that the probabilistic link potential is equivalent to the probabilistic balanced link contributions property. Ortmann (1998) obtains a similar result for TU games.

This article is organized as follows. In Section 2, we introduce notations and preliminaries. Section 3 is devoted to the study of the generalized probabilistic graphs and its subgraphs. The definition of the probabilistic position value can be found in Section 4. In section 5, we present two characterizations of the defined value extending the ones existing for the deterministic case. And finally, the logical relation between these two characterizations is given in Section 6.

2 Preliminaries

A game in characteristic function form (a TU game or a coalitional game) is a pair (N, v), where $N = \{1, 2, \ldots, n\}$ is a finite set of players and v (the characteristic function) is a real map defined on 2^N, the set of all subsets (coalitions) of N, satisfying $v(\emptyset) = 0$. For each $S \subset N$, $v(S)$ represents the value produced by S when its players agree to cooperate. When there is no ambiguity with respect to the set of players N, we identify the game (N, v) with its characteristic function v. For each $S \subset N$, the cardinal of S is denoted by s. The $2^n - 1$ dimensional vector space of all games with players set N is denoted by G^N. A game v in G^N is zero-normalized if $v(\{i\}) = 0$ for all $i \in N$. Let G_0^N denote the subclass of G^N of zero-normalized games.

Shapley (1953) introduces a point solution for the class of cooperative games, widely used in the literature. For each $v \in G^N$, the Shapley value of player $i \in N$ is a convex linear combination of his marginal contributions:

$$Sh_i(N, v) = \sum_{S \subset N,\ i \notin S} \frac{s!(n-1-s)!}{n!} \left[v(S \cup \{i\}) - v(S) \right].$$

This solution is a linear map from G^N on $\mathbb{R}^n$, and thus it can be computed using the so-called unanimity games. Given a coalition $S \subset N$, $S \neq \emptyset$, the unanimity game u_S is defined by $u_S(T) = 1$ if $S \subset T$ and $u_S(T) = 0$ otherwise. The collection of games $\{u_S\}_{\emptyset \neq S \subset N}$ is a basis of G^N, and then, for $v \in G^N$, it holds that:

$$v = \sum_{\emptyset \neq S \subset N} \Delta_v(S) u_S.$$

For each $\emptyset \neq S \subset N$, $\Delta_v(S)$ is known as the Harsanyi dividend (Harsanyi (1959)) of the coalition S and can be obtained from:

$$\Delta_v(S) = \sum_{T \subset S} (-1)^{s-t} v(T).$$

It is easy to see that $Sh_i(u_S) = 1/s$, if $i \in S$ and 0 otherwise. As a consequence, for all $v \in G^N$,

$$Sh_i(N, v) = \sum_{S \subset N,\, i \in S} \frac{\Delta_v(S)}{s}.$$

To formalize social or economic network, we use a graph (N, γ), where $N = \{1, 2, \ldots, n\}$ is the finite set of nodes (actors) and γ a collection of links (edges or ties). A link is an unordered pair $\{i, j\}$, such that $i \neq j$ and $i, j \in N$. A graph (N, δ) is a subgraph of (N, γ) if $\delta \subset \gamma$. When there is no ambiguity with respect to N, we refer to (N, γ) as γ. The complete graph with nodes set N is given by $K_N = \{\{i, j\} \,|\, i \neq j \text{ and } i, j \in N\}$. Denote by 2^{K_N} the class of all networks with nodes set N. Each graph (N, γ) is both an element of 2^{K_N} and a subgraph of K_N.

Two nodes i and j are directly connected in (N, γ) if $\{i, j\} \in \gamma$. If i and j are not directly connected in (N, γ) but there exists a sequence $(i_1 = i, i_2, \ldots, i_k = j)$ such that $\{i_h, i_{h+1}\} \in \gamma$ for $h = 1, \ldots, k-1$, then i and j are connected in (N, γ). A graph (N, γ) is connected if any two nodes $i, j \in N$ are connected. A subset S of N is connected in (N, γ) if the partial graph $(S, \gamma_{|S})$ is connected, where $\gamma_{|S}$ is the set of links of γ of which both incident nodes are in S.

Given a graph (N, γ), the notion of connectivity induces a partition of N in connected components. A connected component is a maximal connected subset. Two distinct nodes i and j are in the same connected component if and only if they are connected. Let N/γ denote the set of all connected components of N in γ and, more generally, for each $S \subset N$, S/γ is the set of all connected components in the partial graph $(S, \gamma_{|S})$. If the connected component to which node i belongs is a singleton, we say that i is an isolated node. Obviously, γ is connected if and only if $|N/\gamma| = 1$. Finally, $L_i(N, \gamma) = \{l \in \gamma | i \in l\}$ denotes the set of links in γ incident on i and $\gamma_{-i} = \gamma \setminus L_i(N, \gamma)$ is the subgraph of γ obtained by removing links in $L_i(N, \gamma)$. When there is no ambiguity with respect to N we write $L_i(\gamma)$ instead of $L_i(N, \gamma)$.

A communication situation is a triplet (N, v, γ), where v is a game in G^N and γ a graph in 2^{K_N}. In a communication situation, the nodes of the graph are the players of N. The only feasible coalitions are formed by players connected in the graph, i.e. the graph introduces the available channels of communication among players.

An allocation rule for communication situations is a function $\Psi : \mathscr{C}^N \to \mathbb{R}^n$, where $\mathscr{C}^N$ is the set of communication situations with players set N. The real number $\Psi_i(N, v, \gamma)$ is the payoff of player i in the game v when the communication possibilities are restricted by the graph γ. Two well known allocation rules for communication situations are the Myerson value and the position value. Given $(N, v, \gamma) \in \mathscr{C}^N$, Myerson (1977) introduces the graph-restricted game (N, v^γ), of which characteristic function v^γ is defined by:

$$v^\gamma(S) = \sum_{T \in S/\gamma} v(T), \quad S \in 2^N,$$

$v^\gamma(S)$ representing the value produced by S when the communication is restricted by γ. The Myerson value, μ, is the Shapley value of this graph-restricted game.

Myerson (1977) characterizes this allocation rule in terms of two appealing properties: component efficiency and fairness. Component efficiency states that the payoffs of the players of a component add up to the worth of this component. The fairness property establishes that the deletion of the link $\{i, j\}$ changes the payoffs of i and j by the same amount. Then Myerson (1980) provides an alternative characterization using component efficiency and the balanced contributions property. This property states that the payoff difference player i experiences if all the links incident to player j are deleted is equal to the payoff difference player j experiences if all the links incident to player i are deleted.

Meessen (1988) and Borm et al. (1992) introduce an alternative associated game for communication situations. Given a communication situation $(N, v, \gamma) \in \mathscr{C}_0^N$, where $\mathscr{C}_0^N$ is the subset of $\mathscr{C}^N$ of the communication situations in which the game is zero-normalized, the link game (γ, r_γ^v) is defined by:

$$r_\gamma^v(\xi) = \sum_{T \in N/\xi} v(T), \quad \xi \subset \gamma.$$

Then, they define the position value, π, as the allocation rule on $\mathscr{C}_0^N$ that equally allocates the Shapley value of each link in the previous game between its two incident nodes, i.e.:

$$\pi_i(N, v, \gamma) = \frac{1}{2} \sum_{l \in L_i(N,\gamma)} Sh_l(\gamma, r_\gamma^v).$$

Borm et al. (1992) provide a characterization of the position value that is valid on the class of communication situations such that the game is zero-normalized and the graph is cycle-free. Slikker (2005a) characterizes this allocation rule using component efficiency and the balanced link contributions property. In the same spirit as Myerson's balanced contributions, the balanced link contributions property states that the payoff difference player i experiences when player j sequentially delete all his links is equal to the payoff difference player j experiences when player i sequentially delete all his links. Slikker (2005b) provides characterizations of the Myeron value and the position value on the class of reward games using potentials. These characterizations are in the same spirit as the corresponding one of Hart and Mas-Colell (1989) for the Shapley value.

Hamiache (1999) introduces another allocation rule that satisfies five specific properties. This allocation rule is not related with our one.

Calvo et al. (1999) extend the model of Myerson by restricting the cooperation in a probabilistic way. They define a probabilistic graph as a pair $(N, \hat{p})$, where $\hat{p}$ is a function that assigns to each link $\{i, j\} \in K_N$ its probability of realization. They assumed the independence of these probabilities and characterized the Myerson value for probabilistic communication situations $(N,\ v,\ \hat{p})$ using the extension of component efficiency, fairness and balanced contributions properties to this new setting.

Gómez et al. (2008) give another step in this direction considering a probability distribution defined on the set of all possible communication graphs 2^{K_N}. For each $\gamma \in 2^{K_N}$, $p(\gamma)$ is the probability of realization of γ. The pair $(N,\ p)$ is referred as a generalized probabilistic graph. A generalized probabilistic communication situation is a triplet $(N,\ v,\ p)$ where v is a cooperative game and $(N,\ p)$ is a generalized probabilistic graph. The class of probabilistic communication situations with player set N is denoted by $\mathscr{G}^N$. Given $(N,\ v,\ p) \in \mathscr{G}^N$, Gómez et al. (2008) define and study the properties of the induced or restricted game $(N,\ v^p)$, where, for each $S \subset N$,

$$v^p(S) = \sum_{\gamma \in 2^{K_N}} p(\gamma) v^\gamma(S)$$

is the expected value of coalition S in this probabilistic framework. They define the generalized probabilistic Myerson value, $\mu(N,\ v,\ p)$, as the Shapley value of the game v^p, and they characterize it using probabilistic extensions of component efficiency, fairness and balanced contributions.

3 Generalized probabilistic graphs

As previously said, a generalized probabilistic graph is a pair $(N,\ p)$ where p is an arbitrary probability function defined over 2^{K_N}. Obviously, p must satisfy $p(\gamma) \geq 0$ for all $\gamma \in 2^{K_N}$ and $\sum_{\gamma \in 2^{K_N}} p(\gamma) = 1$. Denote by $\mathcal{S}_p$ the support of graph (N, p), that is, the set of graphs $\gamma \in 2^{K_N}$ such that $p(\gamma) > 0$, and let $\gamma_p = \cup_{\gamma \in \mathcal{S}_p} \gamma$. The set of all generalized probabilistic graphs with nodes set N is denoted by $\mathscr{P}^N$. When there is no ambiguity with respect to the set N, we sometimes identify $(N,\ p)$ with the probability function p.

The notion of connectivity can be extended to generalized probabilistic graphs in the following way: two nodes i and j in N are directly connected in $(N,\ p)$ when there exists $\gamma \in \mathcal{S}_p$ such that $\{i,\ j\} \in \gamma$. Two nodes i and j are connected in $(N,\ p)$ if there is a sequence of nodes $(i_1 = i,\ i_2,\ \ldots,\ i_k = j)$ such that i_h and i_{h+1} are directly connected in $(N,\ p)$ for all $h = 1,\ 2,\ \ldots,\ k-1$. This notion of connectivity in $(N,\ p)$ induces a partition of N in probabilistic connected components. A probabilistic connected component is a maximal connected set. Denote by N/p the set of all connected components in $(N,\ p)$. Note that $N/p = N/\gamma_p$. Let us observe that two nodes i and j can be in the same connected component of N/p even if there is no $\gamma \in \mathcal{S}_p$ such that i and j are connected in γ.

Each generalized probabilistic graph (N, p) induces $2^{\binom{n}{2}}$ *generalized probabilistic* *subgraphs* $(N, p_\xi) \in \mathscr{P}^N$, one for each $\xi \in 2^{K_N}$, where:

$$p_\xi(\gamma) = \begin{cases} \displaystyle\sum_{\delta \subset K_N \setminus \xi} p(\gamma \cup \delta) = \sum_{\delta \subset \gamma_p \setminus \xi} p(\gamma \cup \delta) & \text{if } \gamma \subset \xi \\ 0 & \text{otherwise.} \end{cases}$$

If $(N,\ p_\xi)$ is a generalized probabilistic subgraph of $(N,\ p)$, we note $(N,\ p_\xi)\widetilde{\subset}(N,\ p)$. The interpretation of $(N,\ p_\xi)$ is the following: only deterministic subgraphs γ of ξ have a strictly positive probability of realization. The probability $p_\xi(\gamma)$ is the sum of the probabilities that p gives to those graphs in 2^{K_N} having in common with ξ the links in γ, i.e.:

$$p_\xi(\gamma) = \begin{cases} \displaystyle\sum_{\eta \in 2^{K_N},\ \eta \cap \xi = \gamma} p(\eta) & \text{if } \gamma \subset \xi \\ 0, & \text{otherwise.} \end{cases}$$

The following proposition provides the condition under which two generalized probabilistic subgraphs of $(N,\ p) \in \mathscr{P}^N$ coincide. This allows to identify the subgraphs that are different.

Proposition 3.1 *For each* $(N,\ p) \in \mathscr{P}^N$ *and each pair* $(N,\ p_\xi)$, $(N,\ p_{\xi'})\widetilde{\subset}(N,\ p)$, *it holds that* $p_\xi = p_{\xi'}$ *if and only if* $\xi \cap \gamma_p = \xi' \cap \gamma_p$.

Proof : Suppose that $p_\xi = p_{\xi'}$ and $\xi \cap \gamma_p \neq \xi' \cap \gamma_p$. Then there exists $l \in \xi \cap \gamma_p$ such that $l \notin \xi'$, or $l \in \xi' \cap \gamma_p$ such that $l \notin \xi$. Without loss of generality, let us consider the former possibility. We have

$$0 = \sum_{\gamma \subset \xi \setminus \{l\}} p_{\xi'}(\gamma \cup \{l\}) = \sum_{\gamma \subset \xi \setminus \{l\}} p_\xi(\gamma \cup \{l\}),$$

where the first equality follows since, for any $\gamma \subset \xi \setminus \{l\}$, $\xi' \not\supset \gamma \cup \{l\}$ and the second one since $p_\xi = p_{\xi'}$. As

$$\sum_{\gamma \subset \xi \setminus \{l\}} p_\xi(\gamma \cup \{l\}) = \sum_{\gamma \subset \xi \setminus \{l\}} \sum_{\delta \subset \gamma_p \setminus \xi} p(\gamma \cup \{l\} \cup \delta) = \sum_{\eta \subset \gamma_p \setminus \{l\}} p(\eta \cup \{l\}),$$

we can conclude that $p(\eta \cup \{l\}) = 0$ for all $\eta \subset \gamma_p \setminus \{l\}$. Then $l \notin \gamma_p$, which is a contradiction.

Reciprocally, to prove that $\xi \cap \gamma_p = \xi' \cap \gamma_p$ implies $p_\xi = p_{\xi'}$, let us show that for each $\xi \subset K_N$, $p_\xi = p_{\xi \cap \gamma_p}$. For any $\gamma \subset \xi \cap \gamma_p$,

$$p_{\xi \cap \gamma_p}(\gamma) = \sum_{\delta \subset K_N \setminus (\xi \cap \gamma_p)} p(\gamma \cup \delta) = \sum_{\delta \subset (K_N \setminus \xi) \cup (K_N \setminus \gamma_p)} p(\gamma \cup \delta) = \sum_{\delta \subset K_N \setminus \xi} p(\gamma \cup \delta) = p_\xi(\gamma)$$

where the third equality follows since, for each δ such that $\delta \cap (K_N \setminus \gamma_p) \neq \emptyset$, $p(\gamma \cup \delta) = 0$. Moreover, if $\gamma \not\subset \xi \cap \gamma_p$, then $p_{\xi \cap \gamma_p}(\gamma) = p_\xi(\gamma) = 0$, which completes the proof. ■

As a direct consequence of previous proposition, we can restrict the family of generalized probabilistic subgraphs of $(N, p) \in \mathscr{P}^N$ to the set $\{(N, p_\xi)\}_{\xi \subset \gamma_p}$, that we denote by $2^{(N,p)}$. In particular, note that (N, p_{K_N}) coincides with (N, p) and (N, p_{γ_p}). Let us extend the standard notation for the deterministic case and denote by $(N, p_{-\xi})$ (or $p_{-\xi}$ when there is no ambiguity on the nodes set N) the probabilistic subgraph $(N, p_{\gamma_p \setminus \xi})$, $\xi \subseteq \gamma_p$. In the special case $\xi = \{l\}$, we will simplify the notation using p_{-l} instead of $p_{-\{l\}}$. Moreover, for all $C \subset N$, we will note $p_C = p_{\gamma_{p|C}}$.

Proposition 3.2 *For each $(N, p) \in \mathscr{P}^N$ and $\xi, \xi' \subset \gamma_p$ such that $\xi \subset \xi'$, it holds that $(N, p_\xi) \widetilde{\subset} (N, p_{\xi'})$.*

Proof : If $\gamma \subset \xi$,

$$\begin{aligned} p_\xi(\gamma) &= \sum_{\delta \subset \gamma_p \setminus \xi} p(\gamma \cup \delta) = \sum_{\alpha \subset \xi' \setminus \xi} \sum_{\eta \subset \gamma_p \setminus \xi'} p(\gamma \cup \alpha \cup \eta) \\ &= \sum_{\alpha \subset \xi' \setminus \xi} p_{\xi'}(\gamma \cup \alpha) = (p_{\xi'})_\xi(\gamma) \end{aligned} \tag{1}$$

where the fourth equality in (1) follows since $\gamma_{p_{\xi'}} = \xi'$. ■

Thus we can define an inclusion relation, noted $\widetilde{\subset}$, in $2^{(N,p)}$, in the following way:

$$\text{for } \xi, \xi' \subset \gamma_p, (N, p_\xi) \widetilde{\subset} (N, p_{\xi'}) \text{ if and only if } \xi \subset \xi'.$$

From this inclusion relation, for each $(N, p) \in \mathscr{P}^N$, the set $2^{(N,p)}$ can be equipped with a Boolean algebra, $(2^{(N,p)}, \widetilde{\cup}, \widetilde{\cap})$. The union of two generalized probabilistic subgraphs (N, p_ξ) and $(N, p_{\xi'})$ of a given (N, p), denoted by $(N, p_\xi) \widetilde{\cup} (N, p_{\xi'})$, is the minimal probabilistic subgraph of (N, p) containing (in the sense of $\widetilde{\subset}$) the probabilistic subgraphs (N, p_ξ) and $(N, p_{\xi'})$, and thus it is equal to $(N, p_{\xi \cup \xi'})$.

In the same way, for all $(N, p_\xi), (N, p_{\xi'}) \widetilde{\subset} (N, p)$, let us define $(N, p_\xi) \widetilde{\cap} (N, p_{\xi'}) = (N, p_{\xi \cap \xi'})$.

As a consequence of the previous definitions, each $(N, p_\xi) \in 2^{(N,p)}$ can be written as $(N, p_\xi) = \widetilde{\cup}_{l \in \xi}(N, p_{\{l\}})$ and thus, every probabilistic subgraph is the union of its *probabilistic links*. In particular, $(N, p) = \widetilde{\cup}_{l \in \gamma_p}(N, p_{\{l\}})$.

The following result is straightforward.

Proposition 3.3 *For each $(N, p) \in \mathscr{P}^N$, $(2^{(N,p)}, \widetilde{\cup}, \widetilde{\cap})$ and $(2^{\gamma_p}, \cup, \cap)$ are isomorphic algebras.*

4 The probabilistic position value

In order to extend the deterministic position value to this new probabilistic setting, we first need to define the corresponding (probabilistic) link game. To this aim, the first question that comes to mind is: who are the players, or the coalitions, in this case? Given $(N, v, p) \in \mathcal{G}_0^N$, the set of probabilistic communication situations with a zero-normalized game, the most natural approach is, in our opinion, to consider as individual players the probabilistic links p_l, $l \in \gamma_p$ and thus, as coalitions, the probabilistic subgraphs $p_\xi \in 2^{(N,p)}$. Obviously, another possible approach is to still consider the links of a particular deterministic graph (the obvious candidate being γ_p) as the players in the probabilistic link game. Fortunately, Proposition 3.3 allows to establish that these two approaches converge if we define properly such games. Recall that in the deterministic case, given a communication situation $(N, v, \gamma) \in \mathcal{C}_0^N$, the associated link game (γ, r_γ^v) is defined by:

$$r_\gamma^v(\xi) = \sum_{T \in N/\xi} v(T) = v^\xi(N), \quad \xi \in 2^\gamma,$$

where (N, v^ξ) is the graph-restricted game (Myerson game) associated to the communication situation (N, v, ξ).

Therefore, using an obvious parallelism, for each $(N, v, p) \in \mathcal{G}_0^N$, we can define the probabilistic link game (p, r_p^v) as follows:

$$r_p^v(p_\xi) = v^{p_\xi}(N), \quad p_\xi \in 2^{(N,p)},$$

where (N, v^{p_ξ}) is the probabilistic Myerson game associated to the probabilistic communication situation (N, v, p_ξ). Moreover,

$$v^{p_\xi}(N) = \sum_{\gamma \subset \xi} p_\xi(\gamma) v^\gamma(N) = \sum_{\gamma \subset \xi} p_\xi(\gamma) r_\xi^v(\gamma),$$

and thus, the defined probabilistic link game (p, r_p^v) can be identified with a (deterministic) "link game" $(\gamma_p, \widehat{r_p^v})$ of which characteristic function is given by:

$$\widehat{r_p^v}(\xi) = \sum_{\gamma \subset \xi} p_\xi(\gamma) r_\xi^v(\gamma), \quad \xi \subset \gamma_p.$$

This is the sense in which we said that this two approaches converge. Nevertheless, we will frequently avoid this identification in order to eliminate misleading effects and thus the probabilistic link game associated to the (probabilistic) communication situation (N, v, p) will be (p, r_p^v).

The following proposition states that the probabilistic link game is component additive.

Proposition 4.1 *Given* $(N, v, p) \in \mathscr{G}_0^N$ *and* $p_\xi \widetilde{\subset} p$*, we have*

$$r_p^v(p_\xi) = \sum_{C \in N/\xi} r_p^v(p_{\xi_{|C}}).$$

Proof : For each $p_\xi \widetilde{\subset} p$, we have:

$$\begin{aligned}
r_p^v(p_\xi) &= \sum_{\gamma \subset \xi} p_\xi(\gamma) r_\xi^v(\gamma) = \sum_{\gamma \subset \xi} p_\xi(\gamma) v^\gamma(N) = \sum_{\gamma \subset \xi} p_\xi(\gamma) \sum_{C \in N/\xi} v^{\gamma_{|C}}(C) \\
&= \sum_{C \in N/\xi} \sum_{\gamma \subset \xi} p_\xi(\gamma) v^{\gamma_{|C}}(C) = \sum_{C \in N/\xi} \left[\sum_{\eta \subset \xi_{|C}} \sum_{\delta \subset (\xi \setminus \xi_{|C})} p_\xi(\eta \cup \delta) \right] v^\eta(C) \\
&= \sum_{C \in N/\xi} \sum_{\eta \subset \xi_{|C}} p_{\xi_{|C}}(\eta) v^\eta(C) = \sum_{C \in N/\xi} r_{p_{\xi_{|C}}}^v(p_{\xi_{|C}}) = \sum_{C \in N/\xi} r_p^v(p_{\xi_{|C}}).
\end{aligned}$$

■

The next result states that the defined probabilistic link game can be identified with a convex linear combination of the associated link games to certain particular (deterministic) communication situations: the corresponding ones to the graphs in the support of the probabilistic graph (N, p). This identification is based on the fact that, for each $(N, p) \in \mathscr{P}^N$, the linear maps [1] $g_p : G_0^N \to G^p$ and $\widehat{g_p} : G_0^N \to G^{\gamma_p}$, respectively defined by $g_p(v) = r_p^v$ and $\widehat{g_p}(v) = \widehat{r_p^v}$, induce, in a natural manner, a linear isomorphism between the respective image sets.

Proposition 4.2 *For all* $(N, v, p) \in \mathscr{G}_0^N$*, it holds that:*

$$\widehat{r_p^v} = \sum_{\gamma \in \mathcal{S}_p} p(\gamma)\, r_{\gamma_p}^v|_\gamma.$$

Proof : For each $\xi \subset \gamma_p$, we have:

$$\begin{aligned}
\sum_{\gamma \in \mathcal{S}_p} p(\gamma)\, r_{\gamma_p}^v|_\gamma(\xi) &= \sum_{\gamma \subset \gamma_p} p(\gamma) r_{\gamma_p}^v(\xi \cap \gamma) = \sum_{\delta \subset \xi} \left[\sum_{\gamma \subset \gamma_p,\, \xi \cap \gamma = \delta} p(\gamma) \right] r_{\gamma_p}^v(\delta) \\
&= \sum_{\delta \subset \xi} p_\xi(\delta) r_{\gamma_p}^v(\delta) = \sum_{\delta \subset \xi} p_\xi(\delta) r_\xi^v(\delta) = \widehat{r_p^v}(\xi)
\end{aligned}$$

which completes the proof. ■

Next, using the introduced probabilistic link game, we can define, in a parallel way to the deterministic case, an alternative allocation rule for generalized probabilistic communication situations, that we will call the probabilistic position value.

[1] In a natural way, we denote by G^p and G^{γ_p} the vector linear spaces of games with players sets p and γ_p respectively.

Definition 4.1 *Given* $(N, v, p) \in \mathscr{G}_0^N$*, the probabilistic position value for player i is defined by:*

$$\pi_i(N, v, p) = \sum_{p_l \in L_i(p)} \frac{1}{2} Sh_{p_l}(p, r_p^v)$$

$L_i(p)$ *being the set of probabilistic links* $\{p_l \widetilde{\subset} p \mid l \in L_i(\gamma_p)\}$.

The following proposition states that the probabilistic position value can be obtained as a convex linear combination of the (deterministic) position values of the appropriated (deterministic) communication situations.

Proposition 4.3 *For each* $(N, v, p) \in \mathscr{G}_0^N$*, it holds that*

$$\pi(N, v, p) = \sum_{\gamma \in \mathcal{S}_p} p(\gamma)\pi(N, v, \gamma).$$

Proof : As the aforementioned isomorphism permits to identify games r_p^v and $\widehat{r_p^v}$, we have, for each $l \in \gamma_p$:

$$Sh_{p_l}(p, r_p^v) = Sh_l(\gamma_p, \widehat{r_p^v})$$

and thus, for each $i \in N$:

$$\begin{aligned}
\pi_i(N, v, p) &= \sum_{p_l \in L_i(p)} \frac{1}{2} Sh_{p_l}(p, r_p^v) = \sum_{l \in L_i(\gamma_p)} \frac{1}{2} Sh_l(\gamma_p, \widehat{r_p^v}) \\
&= \sum_{l \in L_i(\gamma_p)} \frac{1}{2} \sum_{\gamma \in \mathcal{S}_p} p(\gamma)\, Sh_l\,(\gamma_p, r_{\gamma_p}^v|_\gamma) = \sum_{l \in L_i(\gamma)} \frac{1}{2} \sum_{\gamma \in \mathcal{S}_p} p(\gamma)\, Sh_l\,(\gamma, r_\gamma^v) \\
&= \sum_{\gamma \in \mathcal{S}_p} p(\gamma) \sum_{l \in L_i(\gamma)} \frac{1}{2} Sh_l(\gamma, r_\gamma^v) = \sum_{\gamma \in \mathcal{S}_p} p(\gamma)\pi_i(N, v, \gamma),
\end{aligned}$$

where the third equality follows by the linearity of the Shapley value and Proposition 4.2, and the fourth since, for all $l \in \gamma_p \setminus \gamma$, l is a dummy player in the game $(\gamma_p, r_{\gamma_p}^v|_\gamma)$, and, for each $l \in \gamma$, $Sh_l(\gamma_p, r_{\gamma_p}^v|_\gamma) = Sh_l(\gamma, r_\gamma^v)$. ■

5 Characterizations of the probabilistic position value

In this section we present two characterizations of the probabilistic position value which extend the ones existing for the deterministic case. The deterministic position value $\pi : \mathscr{C}_0^N \to \mathbb{R}^n$ is characterized by Slikker (2005a) as the unique allocation rule on $\mathscr{C}_0^N$ satisfying:
i) component efficiency, i.e: for each $C \in N/\gamma$, $\sum_{i \in C} \pi_i(N\, v, \gamma) = v(C)$, and

ii) balanced link contributions, i.e.: for $i,\ j \in N$,

$$\sum_{l\in L_j(\gamma)} [\pi_i(N,\ v,\ \gamma) - \pi_i(N,\ v,\ \gamma \setminus \{l\})] = \sum_{l\in L_i(\gamma)} [\pi_j(N,\ v,\ \gamma) - \pi_j(N,\ v,\ \gamma \setminus \{l\})].$$

The following definitions extend these two properties to the probabilistic case.

Definition 5.1 *An allocation rule $\Psi : \mathcal{G}_0^N \to \mathbb{R}^n$ satisfies component efficiency if, for each $(N,\ v,\ p) \in \mathcal{G}_0^N$ and each $C \in N/p$, $\sum_{i\in C} \Psi_i(N,\ v,\ p) = v^p(C)$ holds.*

Definition 5.2 *An allocation rule $\Psi : \mathcal{G}_0^N \to \mathbb{R}^n$ satisfies the balanced probabilistic link contributions property if, for each $(N,\ v,\ p) \in \mathcal{G}_0^N$ and each $i,\ j \in N$,*

$$\sum_{p_l\in L_i(p)} \left[\pi_j(N,\ v,\ p) - \pi_j(N,\ v,\ p_{-l})\right] = \sum_{p_l\in L_j(p)} \left[\pi_i(N,\ v,\ p) - \pi_i(N,\ v,\ p_{-l})\right].$$

In the next three propositions, it is proved that the defined probabilistic position value is characterized by these two properties.

Proposition 5.1 *The probabilistic position value satisfies component efficiency.*

Proof : Let $(N,\ v,\ p) \in \mathcal{G}_0^N$ and $C \in N/p$. Then:

$$\sum_{i\in C} \pi_i(N,\ v,\ p) = \sum_{i\in C} \sum_{p_l\in L_i(p)} \frac{1}{2} Sh_{p_l}(p,\ r_p^v) = \sum_{i\in C} \sum_{l\in L_i(\gamma_p)} \frac{1}{2} Sh_l(\gamma_p,\ \widehat{r_p^v}). \tag{2}$$

The last term in (2) coincides with:

$$\sum_{l\in\gamma_{p_{|C}}} Sh_l(\gamma_{p_{|C}},\ \widehat{r_{pC}^v}) = \widehat{r_{pC}^v}(\gamma_{p_{|C}}) = v^{p_C}(C) = v^p(C).$$

■

Proposition 5.2 *The probabilistic position value satisfies the balanced probabilistic link contributions property.*

Proof : Given $(N, v, p) \in \mathcal{G}_0^N$ and $i, j \in N$,

$$\sum_{p_l \in L_i(p)} \left[\pi_j(N, v, p) - \pi_j(N, v, p_{-l})\right]$$

$$= \sum_{l \in L_i(\gamma_p)} \left[\sum_{\gamma \in S_p} p(\gamma)\pi_j(N, v, \gamma) - \sum_{\gamma \in S_{p_{-l}}} p_{-l}(\gamma)\pi_j(N, v, \gamma)\right]$$

$$= \sum_{l \in L_i(\gamma_p)} \left[\sum_{\gamma \subset \gamma_p} p(\gamma)\pi_j(N, v, \gamma) - \sum_{\gamma \subset \gamma_p \setminus \{l\}} [p(\gamma \cup \{l\}) + p(\gamma)]\pi_j(N, v, \gamma)\right]$$

$$= \sum_{l \in L_i(\gamma_p)} \left[\sum_{\delta \subset \gamma_p \setminus \{l\}} p(\delta \cup \{l\})\pi_j(N, v, \delta \cup \{l\}) - \sum_{\delta \subset \gamma_p \setminus \{l\}} p(\delta \cup \{l\})\pi_j(N, v, \delta)\right]$$

$$= \sum_{l \in L_i(\gamma_p)} \sum_{\gamma \subset \gamma_p} p(\gamma) \left[\pi_j(N, v, \gamma) - \pi_j(N, v, \gamma \setminus \{l\})\right]$$

$$= \sum_{\gamma \subset \gamma_p} p(\gamma) \sum_{l \in L_i(\gamma_p)} \left[\pi_j(N, v, \gamma) - \pi_j(N, v, \gamma \setminus \{l\})\right]$$

$$= \sum_{\gamma \subset \gamma_p} p(\gamma) \sum_{l \in L_j(\gamma_p)} [\pi_i(N, v, \gamma) - \pi_i(N, v, \gamma \setminus \{l\})]$$

where the last equality follows since the deterministic position value satisfies balanced link contributions. This last term coincides with

$$\sum_{p_l \in L_j(p)} [\pi_i(N, v, p) - \pi_i(N, v, p_{-l})]$$

by the same arguments as for the previous equalities. ■

The proof of the following theorem mimics the corresponding one in Slikker (2005a) and then it is omitted.

Theorem 5.1 *The probabilistic position value is the unique allocation rule on $\mathcal{G}_0^N$ that satisfies component efficiency and balanced probabilistic link contributions.*

Slikker (2005b) provides another characterization of the deterministic position value using a natural extension of the potential function defined by Hart and Mas-Colell (1989) to characterize the Shapley value.

Let P be a real function defined on $\mathcal{C}_0 = \bigcup_{n \in \mathbb{N}} \mathcal{C}_0^N$. The marginal contribution of player i to a communication situation can be defined as the sum of the marginal contributions of each of his incident links:

$$D_i P(N, v, \gamma) = \sum_{l \in L_i(\gamma)} [P(N, v, \gamma) - P(N, v, \gamma \setminus \{l\})]$$

for each $(N, v, \gamma) \in \mathscr{C}_0^N \subset \mathscr{C}_0$ and each $i \in N$.

A function $P : \mathscr{C}_0 \to \mathbb{R}$ is a link potential function if $P(N, v, \emptyset) = 0$ for each $(N, v, \emptyset) \in \mathscr{C}_0$ and the sum of the marginal contributions of players with respect to D equals the value produced by the connected components, i.e.

$$\sum_{i \in N} D_i P(N, v, \gamma) = v^{\gamma}(N)$$

for each $(N, v, \gamma) \in \mathscr{C}_0$ such that $\gamma \neq \emptyset$.

Slikker (2005b) obtains the following result.

Theorem 5.2 *There exists a unique link potential function $P : \mathscr{C}_0 \to \mathbb{R}$. Moreover, for each $(N, v, \gamma) \in \mathscr{C}_0$ and each $i \in N$, $D_i P(N, v, \gamma) = \pi_i(N, v, \gamma)$ holds.*

Let us extend this result to the class of generalized probabilistic communication situations. Let $\mathscr{G}_0 = \bigcup_{n \in \mathbb{N}} \mathscr{G}_0^N$ and Q a real function defined on $\mathscr{G}_0$. The expected marginal contribution of a player to a generalized probabilistic communication situation can be defined as the sum of the expected marginal contributions of each of his incident probabilistic links:

$$M_i Q(N, v, p) = \sum_{p_l \in L_i(p)} [Q(N, v, p) - Q(N, v, p_{-l})] \tag{3}$$

for each $(N, v, p) \in \mathscr{G}_0$ such that $\gamma_p \neq \emptyset$ $(p \neq p_{\emptyset})$ and each $i \in N$.

A function $Q : \mathscr{G}_0 \to \mathbb{R}$ is a probabilistic link potential function if $Q(N, v, p_{\emptyset}) = 0$ for each $(N, v, p_{\emptyset}) \in \mathscr{G}_0$, and the sum of the marginal contributions of players with respect to M equals the expected value produced by the grand coalition, i.e.

$$\sum_{i \in N} M_i Q(N, v, p) = v^{p}(N) \tag{4}$$

for each $(N, v, p) \in \mathscr{G}_0$ such that $p \neq p_{\emptyset}$.

In the following theorem, we show that the probabilistic link potential function is uniquely defined. Moreover, the expected marginal contributions that correspond to this potential coincide with the probabilistic position value.

Theorem 5.3 *There exists a unique probabilistic link potential function $Q : \mathscr{G}_0 \to \mathbb{R}$. Moreover, for each $(N, v, p) \in \mathscr{G}_0$ and each $i \in N$, $M_i Q(N, v, p) = \pi_i(N, v, p)$ holds.*

Proof : Firstly, we show that a probabilistic link potential function exists. Consider

$(N, v, p) \in \mathscr{G}_0$, and define:

$$Q(N, v, p) = \sum_{\xi \subset \gamma_p} \frac{\Delta_{r_p^v}(p_\xi)}{2|\xi|} \tag{5}$$

Obviously, for each $(N, v, p_\emptyset) \in \mathscr{G}_0$ we have $Q(N, v, p_\emptyset) = 0$. Let us prove that Q, as defined in (5), satisfies (4) for each $(N, v, p) \in \mathscr{G}_0$ such that $p \neq p_\emptyset$. It is easy to see that, in this case, for $p_\xi \widetilde{\subset} p_{-l}$, it holds that $r_{p_{-l}}^v(p_\xi) = r_p^v(p_\xi)$. Thus, for each $p_\xi \widetilde{\subset} p_{-l}$, $\Delta_{r_{p_{-l}}^v}(p_\xi) = \Delta_{r_p^v}(p_\xi)$. Then, from (3) and (5), and using the component efficiency property of the probabilistic position value, we obtain:

$$\begin{aligned}
\sum_{i \in N} M_i Q(N, v, p) &= \sum_{i \in N} \sum_{p_l \in L_i(p)} \left(\sum_{\xi \subset \gamma_p} \frac{\Delta_{r_p^v}(p_\xi)}{2|\xi|} - \sum_{\xi \subset \gamma_p \setminus \{l\}} \frac{\Delta_{r_{p_{-l}}^v}(p_\xi)}{2|\xi|} \right) \\
&= \sum_{i \in N} \sum_{p_l \in L_i(p)} \sum_{\xi \subset \gamma_p, l \in \xi} \frac{\Delta_{r_p^v}(p_\xi)}{2|\xi|} = \sum_{i \in N} \frac{1}{2} \sum_{p_l \in L_i(p)} Sh_l(p, r_p^v) \\
&= \sum_{i \in N} \pi_i(N, v, p) = \sum_{C \in N/\gamma_p} v^p(C) = v^p(N).
\end{aligned}$$

Therefore, $Q(N, v, p)$, as defined in (5), is a probabilistic link potential function. Moreover, by the same arguments used to obtain the first four equalities, it holds that $M_i Q(N, v, p) = \pi_i(N, v, p)$ for each $(N, v, p) \in \mathscr{G}_0$ and each $i \in N$.
Secondly, we show that the probabilistic link potential is unique. The combination of (3) and (4) gives:

$$Q(N, v, p) = \frac{1}{2|\gamma_p|} \left(v^p(N) + 2 \sum_{l \in \gamma_p} Q(N, v, p_{-l}) \right).$$

Starting with $Q(N, v, p_\emptyset) = 0$ for each $(N, v, p_\emptyset) \in \mathscr{G}_0$, one can recursively define $Q(N, v, p)$ for each $(N, v, p) \in \mathscr{G}_0$ in a unique way. ■

In next proposition we prove that the probabilistic link potential function admits an expression in terms of the deterministic link potential function.

Proposition 5.3 *The unique probabilistic link potential function $Q : \mathscr{G}_0 \to \mathbb{R}$ satisfies $Q(N, v, p) = \sum_{\gamma \in \mathcal{S}_p} P(N, v, \gamma) p(\gamma)$, where $P : \mathscr{C}_0 \to \mathbb{R}$ is the unique link potential function.*

Proof: To prove this, let us show that $Q'(N, v, p) = \sum_{\gamma \in \mathcal{S}_p} P(N, v, \gamma) p(\gamma)$ is a probabilistic link potential function. By the uniqueness of Q, the result holds. Note that $Q'(N, v, p_\emptyset) = 0$ trivially holds for each $(N, v, p_\emptyset) \in \mathscr{G}_0$. If (N, v, p) is such that $p \neq p_\emptyset$, we have:

$$\begin{aligned}
\sum_{i\in N} M_i Q'(N, v, p) &= \sum_{i\in N} \sum_{p_l\in L_i(p)} [Q'(N, v, p) - Q'(N, v, p_{-l})] \\
&= \sum_{i\in N} \sum_{p_l\in L_i(p)} \left[\sum_{\gamma\in\mathcal{S}_p} P(N, v, \gamma)p(\gamma) - \sum_{\gamma\in\mathcal{S}_{p_{-l}}} P(N, v, \gamma)p_{-l}(\gamma) \right] \\
&= \sum_{i\in N} \sum_{p_l\in L_i(p)} \left[\sum_{\gamma\subset\gamma_p} p(\gamma)P(N, v, \gamma) - \sum_{\gamma\subset\gamma_p\setminus\{l\}} [p(\gamma\cup\{l\}) + p(\gamma)]P(N, v, \gamma) \right] \\
&= \sum_{i\in N} \sum_{p_l\in L_i(p)} \left[\sum_{\delta\subset\gamma_p\setminus\{l\}} p(\delta\cup\{l\})P(N, v, \delta\cup\{l\}) - \sum_{\delta\subset\gamma_p\setminus\{l\}} p(\delta\cup\{l\})P(N, v, \delta) \right] \\
&= \sum_{i\in N} \sum_{p_l\in L_i(p)} \sum_{\gamma\subset\gamma_p} p(\gamma) [P(N, v, \gamma) - P(N, v, \gamma\setminus\{l\})] \\
&= \sum_{\gamma\subset\gamma_p} p(\gamma) \sum_{i\in N} \sum_{l\in L_i(\gamma_p)} [P(N, v, \gamma) - P(N, v, \gamma\setminus\{l\})] \\
&= \sum_{\gamma\subset\gamma_p} p(\gamma)v^\gamma(N) = v^P(N),
\end{aligned}$$

where the seventh equality follows since P is the unique potential link function for the deterministic case. ■

6 On the relation between the potential and the balanced probabilistic link contributions property

In this section, we prove the equivalence between the two previous characterizations. Before providing the main result of this section, we introduce two more definitions and highlight a property of the probabilistic link potential function.

In a communication situation, players who are not in the same connected component are not able to coordinate their actions. Then, the connected components operate independently: there is no externalities between them. A function $Q : \mathcal{G}_0 \to \mathbb{R}$ satisfies component additivity if it captures this idea.

Definition 6.1 *A function $Q : \mathcal{G}_0 \to \mathbb{R}$ is component additive if for each $(N, v, p) \in \mathcal{G}_0$,*

$$Q(N, v, p) = \sum_{T\in N/\gamma_p} Q(T, v_{|T}, p_T).$$

In the next proposition, we state that the probabilistic link potential satisfies this property.

Proposition 6.1 *The probabilistic link potential is component additive.*

Proof : Let us recall that, for a deterministic communication situation $(N, v, \gamma) \in \mathscr{C}_0$, the dividends $\Delta_{r^v_\gamma}(\xi)$ are equal to zero if ξ is not connected (see, for example, Gómez et al. (2004)). Given a graph (N, γ) and $S \subseteq N$, let us note $S(\gamma) = \{i \in S \text{ such that } \exists\ l \in \gamma \text{ with } i \in l\}$ the set of not isolated players of S in the graph (N, γ). Thus, if $P : \mathscr{C}_0 \to \mathbb{R}$ is the deterministic link potential function, we have:

$$P(N,\ v,\ \gamma) = \sum_{\xi \subset \gamma} \frac{\Delta_{r^v_\gamma}(\xi)}{2|\xi|} = \sum_{T \in N/\gamma} \sum_{\substack{\xi \subset \gamma_{|T} \\ |N(\xi)/\xi|=1}} \frac{\Delta_{r^v_\gamma}(\xi)}{2|\xi|} = \sum_{T \in N/\gamma} \sum_{\substack{\xi \subset \gamma_{|T} \\ |N(\xi)/\xi|=1}} \frac{\Delta_{r^{v_{|T}}_{\gamma_{|T}}}(\xi)}{2|\xi|} = \sum_{T \in N/\gamma} P(T,\ v_{|T},\ \gamma_{|T}).$$

Then P is component additive. For the probabilistic link potential function Q, we have,

$$\begin{aligned}
Q(N,\ v,\ p) &= \sum_{\gamma \in \mathcal{S}_p} p(\gamma) P(N,\ v,\ \gamma) = \sum_{\gamma \in \mathcal{S}_p} p(\gamma) \sum_{T \in N/\gamma} P(T,\ v_{|T},\ \gamma_{|T}) \\
&= \sum_{C \in N/\gamma_p} \sum_{\gamma \in \mathcal{S}_p} p(\gamma) \sum_{T \in N/\gamma_{|C}} P(T,\ v_{|T},\ \gamma_{|T}) \\
&= \sum_{C \in N/\gamma_p} \sum_{\gamma \subset \gamma_p} p(\gamma) P(C,\ v_{|C},\ \gamma_{|C}) \\
&= \sum_{C \in N/\gamma_p} \sum_{\substack{\gamma = \delta \cup \xi \\ \delta \subset \gamma_{|C} \\ \xi \subset \gamma_p \setminus \gamma_{|C}}} p(\gamma) P(C,\ v_{|C},\ \gamma_{|C}) \\
&= \sum_{C \in N/\gamma_p} \sum_{\delta \subset \gamma_{|C}} p_C(\delta) P(C,\ v_{|C},\ \gamma_{|C}) \\
&= \sum_{C \in N/\gamma_p} Q(C,\ v_{|C},\ \gamma_{|C}).
\end{aligned}$$

■

An allocation rule is component decomposable if the payoff of a player is not affected by the values created by players who do not belong to his component.

Definition 6.2 *An allocation rule $\Psi : \mathscr{G}_0 \to \mathbb{R}$ is component decomposable if, for each $(N,\ v,\ p) \in \mathscr{G}_0$ and each $i \in N$,*

$$\Psi_i(N,\ v,\ p) = \Psi_i(T_i,\ v_{|T_i},\ p_{T_i}),$$

where T_i is the connected component of N/p to which i belongs.

In the following theorem we prove that the balanced link contributions property and the variations used to define the probabilistic link potential are equivalent.

Theorem 6.1 *Let Ψ be a component decomposable allocation rule on $\mathscr{G}_0$ verifying $\Psi_i(N,\ v,\ p_\emptyset) = 0$ for each $(N,\ v,\ p_\emptyset) \in \mathscr{G}_0$ and each $i \in N$. This allocation rule* *satisfies balanced probabilistic link contributions on $\mathscr{G}_0$ if and only if there exists a component additive function $Q : \mathscr{G}_0 \to \mathbb{R}$ such that $Q(N, v, p_\emptyset) = 0$ for each $(N, v, p_\emptyset) \in \mathscr{G}_0$ and $M_iQ(N, v, p) = \Psi_i(N, v, p)$ for each $(N, v, p) \in \mathscr{G}_0$ with $p \neq p_\emptyset$ and each $i \in N$, $M_iQ(N, v, p)$ being defined as in (3).*

Proof: Suppose that a component additive function $Q : \mathscr{G}_0 \to \mathbb{R}$ satisfying $Q(N, v, p_\emptyset) = 0$ for each $(N, v, p_\emptyset) \in \mathscr{G}_0$ and $M_iQ(N, v, p) = \Psi_i(N, v, p)$ for each $(N, v, p) \in \mathscr{G}_0$ with $p \neq p_\emptyset$ and each $i \in N$ exists. Let us prove that Ψ verifies balanced probabilistic link contributions. We have:

$$\begin{aligned}
\sum_{p_l \in L_j(p)} [\Psi_i(N, v, p) - \Psi_i(N, v, p_{-l})] &= \sum_{p_l \in L_j(p)} [M_iQ(N, v, p) - M_iQ(N, v, p_{-l})] \\
&= \sum_{p_l \in L_j(p)} \Bigg[\sum_{p_k \in L_i(p)} [Q(N, v, p) - Q(N, v, p_{-k})] \\
&\quad - \sum_{p_k \in L_i(p)} [Q(N, v, p_{-l}) - Q(N, v, p_{-\{k,l\}})] \Bigg] \\
&= \sum_{p_l \in L_j(p)} \sum_{p_k \in L_i(p)} [Q(N, v, p) - Q(N, v, p_{-k})] \\
&\quad - \sum_{p_l \in L_j(p)} \sum_{p_k \in L_i(p)} [Q(N, v, p_{-l}) - Q(N, v, p_{-\{k,l\}})] \\
&= \sum_{p_k \in L_i(p)} [\Psi_j(N, v, p) - \Psi_j(N, v, p_{-k})].
\end{aligned}$$

Note that the expression after the third equality sign is symmetric in i and j. The last equality follows by the same arguments as for the first three ones.
Conversely, let Ψ be a component decomposable allocation rule on $\mathscr{G}_0$ verifying balanced probabilistic link contributions and such that $\Psi_i(N, v, p_\emptyset) = 0$ for each $(N, v, p_\emptyset) \in \mathscr{G}_0$ and each $i \in N$. We have to show that we can define a component additive function $Q : \mathscr{G}_0 \to \mathbb{R}$, such that $Q(N, v, p_\emptyset) = 0$ for each $(N, v, p_\emptyset) \in \mathscr{G}_0$ and $M_i(N, v, p) = \Psi_i(N, v, p)$ for each $(N, v, p) \in \mathscr{G}_0$ with $p \neq p_\emptyset$ and each $i \in N$. We proceed by induction on $|\gamma_p|$.
Firstly, note that this assertion is trivially verified for $|\gamma_p| = 0$. Then, by induction hypothesis, let us assume that there exists a component additive function Q such that $Q(N, v, p_\emptyset) = 0$ for each $(N, v, p_\emptyset) \in \mathscr{G}_0$ and $M_iQ(N, v, p) = \Psi_i(N, v, p)$ for each $(N, v, p) \in \mathscr{G}_0$ with $p \neq p_\emptyset$, $|\gamma_p| < k$ and each $i \in N$. Consider $(N, v, p) \in \mathscr{G}_0$ such that $|\gamma_p| = k$. Remark that if $T \in N/\gamma_p$, $T = \{i\}$, $Q(\{i\}, v_{|\{i\}}, p_{\{i\}}) = 0$. Consider now $T \in N/\gamma_p$ such that $|T| > 1$. Note that, for each $i \in T$, $|L_i(p)| \neq 0$ and $L_i(p_T) = L_i(p)$. Let us define:

$$Q(T, v_{|T}, p_T) = \frac{\Psi_i(T, v_{|T}, p_T) + \sum_{p_{l_i} \in L_i(p)} Q(T, v_{|T}, (p_T)_{-l_i})}{|L_i(p)|}$$

where i is whatever element in T. We relegate to the Appendix the proof that $Q(T,\ v_{|T},\ p_T)$ is well defined in the sense that it does not depends on $i \in T$. Then, we define

$$Q(N,\ v,\ p) = \sum_{T \in N/p} Q(T,\ v_{|T}, p_T).$$

Consider $i \in N$. By the induction hypothesis, we have:

$$\begin{aligned} M_i Q(N,\ v,\ p) &= \sum_{p_{l_i} \in L_i(p)} [Q(N,\ v,\ p) - Q(N,\ v,\ p_{-l_i})] \\ &= \sum_{p_{l_i} \in L_i(p)} \left[\sum_{T \in N/p} Q(T,\ v_{|T}, p_T) - \sum_{T \in N/p_{-l_i}} Q(T,\ v_{|T}, (p_T)_{-l_i}) \right]. \end{aligned}$$

For each $T \in N/p$ such that $i \notin T$, $Q(T,\ v_{|T}, p_T) = Q(T,\ v_{|T}, (p_T)_{-l_i})$. Denote by T_i the connected component of N/p containing i. It holds that:

$$\begin{aligned} M_i Q(N,\ v,\ p) &= \sum_{p_{l_i} \in L_i(p)} [Q(T_i,\ v_{|T_i},\ p_{T_i}) - \sum_{S \in T_i/(p_{T_i})_{-l_i}} Q(S,\ v_{|S},\ (p_S)_{-l_i})] \\ &= \sum_{p_{l_i} \in L_i(p)} [Q(T_i,\ v_{|T_i},\ p_{T_i}) - Q(T_i,\ v_{|T_i},\ (p_{T_i})_{-l_i})]\,. \end{aligned}$$

By (6) and by component decomposability of Ψ, we obtain:

$$M_i Q(N,\ v,\ p) = \Psi_i(T_i,\ v_{|T_i},\ p_{T_i}) = \Psi_i(N,\ v,\ p).$$

■

Remark 6.1 *Note that the definition of Slikker's (2005b) deterministic player potential can be generalized to the probabilistic case in the following way. Denote by $Q : \mathscr{G} \to \mathbb{R}$ the function that assigns to every generalized probabilistic communication situation in $\mathscr{G}$ a real number. The expected marginal contribution of a player to a generalized probabilistic communication situation can be defined as the expected marginal contribution of all his links:*

$$\mathcal{M}_i P(N,\ v,\ p) = Q(N,\ v,\ p) - Q(N,\ v,\ p_{-i}) \tag{6}$$

for each $(N,\ v,\ p) \in \mathscr{G}$ and each $i \in N$, p_{-i} being the generalized probabilistic subgraph $p_{-L_i(\gamma_p)}$.

A function $Q : \mathscr{G} \to \mathbb{R}$ is a probabilistic player potential function if the sum of the marginal contributions of players with respect to $\mathcal{M}$ is equal to the expected value produced by the grand coalition, i.e. $Q(N,\ v,\ p_\emptyset) = 0$ for each $(N,\ v,\ p_\emptyset) \in \mathscr{G}$ and

$$\sum_{i \in N} \mathcal{M}_i Q(N,\ v,\ p) = v^p(N) \tag{7}$$

for each $(N, v, p) \in \mathscr{G}$ such that $p \neq p_\emptyset$. Results in Theorems 5.3 and 6.1 can be provided for the probabilistic Myerson value of Gómez et al. (2008) and the probabilistic player potential.

Appendix

Suppose Ψ is a component decomposable allocation rule defined on $\mathscr{G}_0$ satisfying balanced probabilistic link contributions and such that $\Psi_i(N, v, p_\emptyset) = 0$ for each $(N, v, p_\emptyset) \in \mathscr{G}_0$ and each $i \in N$. Let us prove that, under these hypothesis, given $(N, v, p) \in \mathscr{G}_0$, $T \in N/p$ with $|T| > 1$ and $i \in T$, $Q(T,\ v_{|T},\ p_T)$, as given in (6) is well defined, i.e., it does not depend on $i \in T$.

Proof : Consider $i,\ j \in T$. By balanced probabilistic link contributions and component decomposability of Ψ,

$$\Psi_i(T,\ v_{|T},\ p_T) = \frac{\sum_{p_{l_j} \in L_j(p)} \Psi_i(T,\ v_{|T},\ (p_T)_{-l_j})}{|L_j(p)|} + \frac{|L_i(p)|\Psi_j(T,\ v_{|T},\ p_T)}{|L_j(p)|} - \frac{\sum_{p_{l_i} \in L_i(p)} \Psi_j(T,\ v_{|T},\ (p_T)_{-l_i})}{|L_j(p)|}.$$

Substituting this previous expression in the right-hand term of (6) we obtain:

$$\frac{1}{|L_i(p)|}\left[\frac{\sum_{p_{l_j} \in L_j(p)} \Psi_i(T,\ v_{|T},\ (p_T)_{-l_j})}{|L_j(p)|} + \frac{|L_i(p)|\Psi_j(T,\ v_{|T},\ p_T)}{|L_j(p)|} - \frac{\sum_{p_{l_i} \in L_i(p)} \Psi_j(T,\ v_{|T},\ (p_T)_{-l_i})}{|L_j(p)|} + \sum_{p_{l_i} \in L_i(p)} Q(T,\ v_{|T},\ (p_T)_{-l_i})\right]. \quad (8)$$

Moreover, by the induction hypothesis in the proof of Theorem 6.1., we know that:

$$\Psi_i(T,\ v_{|T},\ (p_T)_{-l_j}) = |L_i(p)\backslash\{p_{l_j}\}|Q(T,\ v_{|T},\ (p_T)_{-l_j}) - \sum_{p_{l_i} \in L_i(p)\backslash\{p_{l_j}\}} Q(T,\ v_{|T},\ (p_T)_{-\{l_i, l_j\}}). \quad (9)$$

Suppose that i and j have no probabilistic link in common. In this case, (9) can be written as:

$$\Psi_i(T,\ v_{|T},\ (p_T)_{-l_j}) = |L_i(p)|Q(T,\ v_{|T},\ (p_T)_{-l_j}) - \sum_{p_{l_i} \in L_i(p)} Q(T,\ v_{|T},\ (p_T)_{-\{l_i, l_j\}}). \quad (10)$$

Using (10) we have that (8) is equal to:

$$\frac{\Psi_j(T,\ v_{|T},\ p_T)}{|L_j(p)|} + \frac{1}{|L_i(p)|}\left[\frac{\sum_{p_{l_j}\in L_j(p)} |L_i(p)|Q(T,\ v_{|T},(p_T)_{-l_j})}{|L_j(p)|}\right.$$

$$-\frac{\sum_{p_{l_j}\in L_j(p)}\sum_{p_{l_i}\in L_i(p)} Q(T,\ v_{|T},\ (p_T)_{-\{l_i,l_j\}})}{|L_j(p)|} - \frac{\sum_{p_{l_i}\in L_i(p)} |L_j(p)|Q(T,\ v_{|T},(p_T)_{-l_i})}{|L_j(p)|}$$

$$\left.+\frac{\sum_{p_{l_i}\in L_i(p)}\sum_{p_{l_j}\in L_j(p)} Q(T,\ v_{|T},(p_T)_{-\{l_i,l_j\}})}{|L_j(p)|} + \sum_{p_{l_i}\in L_i(p)} Q(T,\ v_{|T},(p_T)_{-l_i})\right]$$

$$= \frac{\Psi_j(T,\ v_{|T},p_T) + \sum_{p_{l_j}\in L_j(p)} Q(T,\ v_{|T},(p_T)_{-l_j})}{|L_j(p)|}.$$

Finally, as the right-hand side in (6) coincides with (8) and thus with the second term in previous equality, we have:

$$\frac{\Psi_i(T,\ v_{|T},\ p_T) + \sum_{p_{l_i}\in L_i(p)} Q(T,\ v_{|T},\ (p_T)_{-l_i})}{|L_i(p)|} = \frac{\Psi_j(T,\ v_{|T},\ p_T) + \sum_{p_{l_j}\in L_j(p)} Q(T,\ v_{|T},\ (p_T)_{-l_j})}{|L_j(p)|}$$

and so the result is proved for the case in which $i, j \in T$ have no probabilistic link in common.

Now, suppose that i and j have a probabilistic link in common. Thus, if $l_j \neq \{i,\ j\}$, (9) can be written as:

$$\Psi_i(T,\ v_{|T},\ (p_T)_{-l_j}) = |L_i(p)|Q(T,\ v_{|T},\ (p_T)_{-l_j}) - \sum_{p_{l_i}\in L_i(p)} Q(T,\ v_{|T},\ (p_T)_{-\{l_i,l_j\}}). \quad (11)$$

and if $l_j = \{i,\ j\}$,

$$\Psi_i(T,\ v_{|T},\ (p_T)_{-\{i,j\}}) = (|L_i(p)|-1)Q(T,\ v_{|T},\ (p_T)_{-\{i.j\}}) - \sum_{p_{l_i}\in L_i(p)\setminus\{p_{\{i,j\}}\}} Q(T,\ v_{|T},\ (p_T)_{-\{l_i,\{i,j\}\}}). \quad (12)$$

Then, substituting (11) and (12) in (8) we have that (8) is equal to:

$$\frac{\Psi_j(T,\ v_{|T}, p_T)}{|L_j(p)|} + \frac{1}{|L_i(p)|}\left[\frac{\sum\limits_{\substack{p_{l_j}\in L_j(p)\\ l_j\neq\{i,j\}}} |L_i(p)|Q(T,\ v_{|T}, (p_T)_{-l_j})}{|L_j(p)|}\right.$$

$$-\frac{\sum\limits_{\substack{p_{l_j}\in L_j(p)\\ l_j\neq\{i,j\}}} \sum\limits_{p_{l_i}\in L_i(p)} Q(T,\ v_{|T}, (p_T)_{-\{l_i, l_j\}})}{|L_j(p)|} + \frac{(|L_i(p)| - 1)Q(T,\ v_{|T}, (p_T)_{-\{i, j\}})}{|L_j(p)|}$$

$$-\frac{\sum\limits_{\substack{p_{l_i}\in L_i(p)\\ l_i\neq\{i,j\}}} Q(T,\ v_{|T}, (p_T)_{-\{\{i, j\}, l_i\}})}{|L_j(p)|} - \frac{\sum\limits_{\substack{p_{l_i}\in L_i(p)\\ l_i\neq\{i,j\}}} |L_j(p)|Q(T,\ v_{|T}, (p_T)_{-l_i})}{|L_j(p)|}$$

$$+\frac{\sum\limits_{\substack{p_{l_i}\in L_i(p)\\ l_i\neq\{i,j\}}} \sum\limits_{p_{l_j}\in L_j(p)} Q(T,\ v_{|T}, (p_T)_{-\{l_i, l_j\}})}{|L_j(p)|} - \frac{(|L_j(p)| - 1)Q(T,\ v_{|T}, (p_T)_{-\{i, j\}})}{|L_j(p)|}$$

$$\left.+\frac{\sum\limits_{\substack{p_{l_j}\in L_j(p)\\ l_j\neq\{i,j\}}} Q(T,\ v_{|T}, (p_T)_{-\{l_j, \{i, j\}\}})}{|L_j(p)|} + \sum_{p_{l_i}\in L_i(p)} Q(T,\ v_{|T}, (p_T)_{-\{l_i\}})\right].$$

And after some straightforward calculations we obtain that previous expression, and then (8), equals to:

$$
\begin{aligned}
&\frac{\Psi_j(T,\, v_{|T}, p_T)}{|L_j(p)|} + \frac{1}{|L_i(p)|}\left[\frac{\sum\limits_{p_{l_j}\in L_j(p)} |L_i(p)| Q(T,\, v_{|T}, (p_T)_{-l_j})}{|L_j(p)|} \right. \\
&- \frac{\sum\limits_{\substack{p_{l_j}\in L_j(p)\\ l_j\neq\{i,\,j\}}} \sum\limits_{p_{l_i}\in L_i(p)} Q(T,\, v_{|T}, (p_T)_{-\{l_i,l_j\}})}{|L_j(p)|} - \frac{\sum\limits_{\substack{p_{l_i}\in L_i(p)\\ l_i\neq\{i,\,j\}}} Q(T,\, v_{|T}, (p_T)_{-\{l_i,\{i,\,j\}\}})}{|L_j(p)|} \\
&- \frac{\sum\limits_{p_{l_i}\in L_i(p)} |L_j(p)| Q(T,\, v_{|T}, (p_T)_{-l_i})}{|L_j(p)|} + \frac{\sum\limits_{\substack{p_{l_i}\in L_i(p)\\ l_i\neq\{i,\,j\}}} \sum\limits_{p_{l_j}\in L_j(p)} Q(T,\, v_{|T}, (p_T)_{-\{l_i,l_j\}})}{|L_j(p)|} \\
&\left. + \frac{\sum\limits_{\substack{p_{l_j}\in L_j(p)\\ l_j\neq\{i,\,j\}}} Q(T,\, v_{|T}, (p_T)_{-\{l_j,\{i,\,j\}\}})}{|L_j(p)|} + \sum_{p_{l_i}\in L_i(p)} Q(T,\, v_{|T}, (p_T)_{-l_i}) \right].
\end{aligned}
$$

Note that in previous expression the sum of the second to sixth terms inside the brackets vanishes and so, the result is proved. ■

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